The Physics of Ocean Waves (for physicists and surfers)

Michael Twardos: *mtwardos@uci.edu*

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Even before I was a physics major on the icy banks of the Raritan in New Jersey, I was fascinated with the ocean; where the waves came from, sand dune formation, tide forecasting, etc. As an undergraduate, I was able to learn about physical theories as abstract as the theory of quantum electrodynamics and color symmetries of quarks. Learning about the dynamics of the ocean, however, was far more elusive. Physics graduate school took me to the University of California Irvine, less that 30 miles from one of the best surfspots (Trestles) in the continental United States! During this time, I managed to surf often while still pondering about how the ocean works. Access to information about this system still seemed to remain out of reach. By my fourth year in graduate school when I was already nearing the end of my research on the flow of foams and sand (and other complex fluids) I finally started learning some fluid dynamics (Navier Stokes, instabilities and nonlinear waves) and other approximations in continuum mechanics. I got my hands on a couple of good books on fluid dynamics, ocean waves and physical oceanography. My perspective on the ocean finally began to expand a little wider!

What follows is the result. This text is intended to be a general but comprehensive overview on physical phenomena associated with the marine environment. I provide my own perspective on this field from my particular experiences and training. Throughout your reading, you may find that I have occasionally strayed rather far that what most people would consider in the bounds of the "physics of ocean waves" (and surfing). There are, indeed, alot of great physical systems that have similarities to the ocean and they are all wrapped up in the subtleties of geom-

Figure 1: On longer period northern swells, the waves wrap around the point at Rincon sucking through the deep underwater canyons and making some great surf.

etry, symmetries, conservation laws and statistics. You'll have to forgive me for mentioning them too.

1 Introduction: The Science of the Ocean

"Our imagination is stretched to the utmost, not as in fiction to imagine things which are not really there but just to comprehend those things which are there."

-Richard Feynman

The ocean is a place of inspiration and mystery. I don't think it's given the attention or appreciation it deserves for inspiring scientific reasoning (as well as artistic inspiration and cultural influence). Physicists in particular should be very thankful for being able to witness the dynamics of macroscopic fluid flow that can be seen in the ocean. What other way is there to conceptualize a travelling wave, the basis for so many fields of classical and modern physics, than by watching the ocean? Where would physicists be in their understanding of flux, continuity or conservation laws without observing the dynamics of fluids? What system is easier way visualize the effects of nonlinearity than from soliton waves found in bodies of water or the instabilities which create the turbulence of choppy seas? As a result of this under appreciation, I am writing to pay my respects to the ocean. Ocean dynamics are the main subject of this book and a major inspiration for my career in physics.

People are naturally amazed by ocean waves, (just turn on a surf video at a party to see what I mean!). We can spend hours mesmerized by the repetitive yet surprising behavior of each incoming set. Not only do we watch the unique dynamics of the water where the ocean meets the land, we also look out over the horizon and ponder "Where did they come from?" "How were they made?" and "Why do they break like that?!" Some of the people most attuned to the fantastic phenomenon of

Figure 2: One of the best waves in the world: Honolua Bay, Maui.

the breaking wave is the surfer. Surfers spend their days and their lifetimes, waiting for their next wave, endlessly seeking the "perfect wave". In my own experiences surfing waves and peering down the blue wall, I seem to always be asking myself, "How does the water do that?!!?" I am still not sure. Surfing seems to be surreal What can be understood about ocean waves? What is still left to be known?

At the beach, one can witness many interesting phenomena besides the waves. The weather or geological formation of the cliffs and sand dunes are immediately apparent, while the patterns of the sand ripples or the origin of the ocean currents are more subtle. Coastlines are a special place on our planet where the land, the atmosphere and the ocean all meet. It is the place to witness fascinating physical phenomena. There may be no other place in our solar system or the entire universe as unique as the coastlines on our planet. Besides reviewing some of our physical understanding of ocean waves, these other patterns that can be found on the coast will also be explored to make this text more complete. In this work I will only be able to review progress on these phenomenon at a basic level. One must realize although much has been achieved in fluid dynamics and physical oceanography, there is still alot left to be studied and it is an intense current area of research!

Richard Feynman was a great physicist of the 20th century who was able to explain complicated physical theories to students and the public with ingenious simplicity. He talked about the fear people have in discussing something beautiful with the language of science. He justified science by trying to show that we can still appreciate apparent beauty while unlocking beauty further hidden beneath the surface. In this work, my last hope is to demystify the wonder of the ocean, the beach and the waves. Instead, I hope to explore these phenomena to appreciate them more and maybe, even uncover something that we did not initially notice.

Although I hope that this text may serve as a reference guide for any with adequate mathematical skill, some more advanced level mathematics may be encountered. Several mathematical languages will be reviewed. Things assumed in this text include basic calculus, integrals, derivatives and some working knowledge of differential equations and vector calculus. I hope to share with you some of the excitement and the beauty of the ocean and studying these systems with the aid of mathematics.

2 Wave Mechanics

In short, physics has discovered that there are no solids, no continuous surfaces, no straight lines, only waves."

Buckminster Fuller (Intuition: Metaphysical Mosaic)

Hawaiins like to say that everything in Hawaii sways back and forth; the waves rise up and break, the tides go in and out, the palm trees sway back and forth and the hoola girls dance from side to side. Indeed, swaying "back and forth" is a fundamental feature in nature. Oscillations or waves are a primary solution to the differential equations that govern many physical phenomena.

This chapter will review differential equations, using the the wave equation as the primary example. Differential equations can be considered the rules by which the universe operates. They describe a balance between the rate of change between different 'observables'. Differential equations are troublesome to learn because there are no standard methods to solve them. You can always attempt a solution by the power series method and match up the coefficients or you can numerically solve many equations on the computer. It just so happens that the overall best way to solve differential equations is by guessing or looking up the way it was already solved by someone who usually "guessed" the answer. It has been working for mathematicians for centuries now!

Other than finding the solution to differential equations, it is often very difficult to write them down in the first place. In order to write down a relationship between variables, one must determine the ways in which derivatives of different variables relate to each other. Likewise, it is also a major challenge to write down the correct boundary conditions (we will do this later) and implement them correctly to "guess" the correct answer.

Regardless, this section will give a brief overview of wave phenomena and the associated mathematical formulation of differential equations. The reader must be reminded that this is not a comprehensive introduction to partial differential equations or classical mechanics. It is meant only as an introductory review to establish the nature of waves that will be extended in future paragraphs.

2.1 The Harmonic Oscillator

The harmonic oscillator may very well be the most important equation in all of physics and differential equations. Suppose we have a mass on a spring to which an external force is applied. The setup is shown in Figure 2. The force that the spring responds with is proportional to the distance it is displaced from equilibrium. We can write this as

$$F = -kx \tag{1}$$

where k is the "spring constant" or proportionality between the force F and the displacement x. This is often referred to as Hooke's Law.

We also know from Newton that there is an inertial force acting on an object whenever it is accelerating. This can be expressed in the form

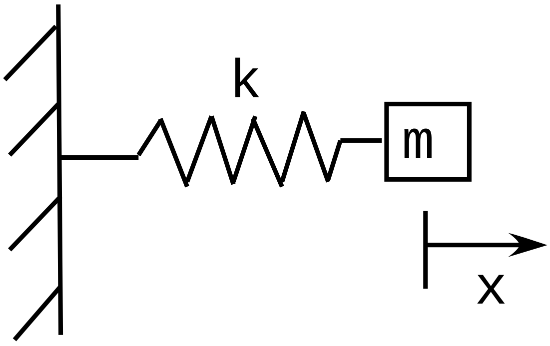

Figure 3: This is a harmonic oscillator. The dynamics of the system are driven by the force of the spring as well as by the mass's inertia.

$$F = m\frac{\partial^2 x}{\partial t^2} \tag{2}$$

These two equations can be combined to write:

$$m\frac{\partial^2 x}{\partial t^2} = -kx \tag{3}$$

This is the equation of the simple harmonic oscillator. Its solution of course is

$$x(t) = A\sin(\sqrt{\frac{k}{m}}t + \phi) \tag{4}$$

where A is the amplitude of the wave and ϕ is the phase factor. Try it out for yourself, differentiate it twice and see if the two sides equal each other. A sine wave describes an oscillation. The solutions is plotted in Figure 3. With no dampening or dissipation the mass attached to the spring will sway back and forth forever. This is our first example of wavelike motion. One thing to note about this equation is that it is linear. This means that the "operator" acting on the variable of interest

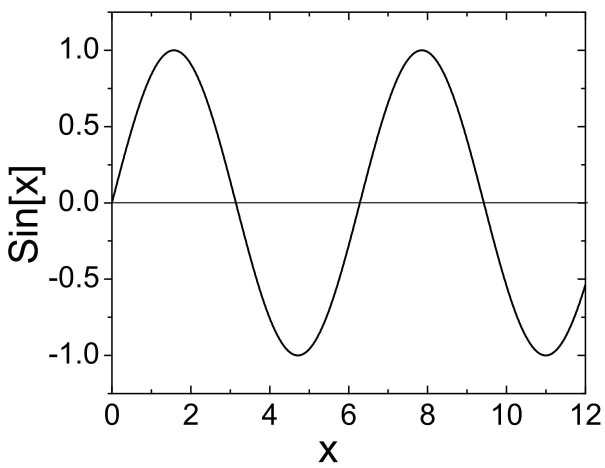

Figure 4: This is a sine wave.

does not, itself, contain that variable of interest. Also, this means, that if we can find two solutions to the equation, then a "linear combination of those solutions is also a solution.

We can write that the "angular frequency" as

$$\omega = \sqrt{\frac{k}{m}} \tag{5}$$

Defining the angular frequency of an oscillation is very useful. It is related to the actual frequency, f, of an oscillation by

$$\omega = 2\pi f \tag{6}$$

which tells you how often something happens and has units of inverse time. To go full circle in this description, the period "T" of an oscillation is the time it takes for one full cycle to be complete. It is related to the angular frequency as:

$$T = \frac{2\pi}{\omega} \tag{7}$$

This discussion only touches on the most important elements of the linear differential equation of the harmonic oscillator. We could go ALOT deeper into the mechanics of this system by considering what happens when it is driven by an external force or damped. However, for our cases, this is not necessary. Let's just say that for anything swaying back and forth, whatever it may be, it can first be approximated as a linear harmonic oscillator.

2.2 The Wave Equation and its Solutions

The harmonic oscillator differential equation has derivatives with respect to only one variable, time. In contrast, the key defining property of a partial differential equation is that there is more than one independent variable x, y, t.... We can say that the "order" of a partial differential equation is the highest derivative that appears. The wave equation is one of the most popular PDE's that describes how the change in space of a wave is related to a change in time. The wave equation can be derived by considering small perturbations in, for example, a guitar string. In this instance, the force (density times the second derivative of the function with respect to time) is related to the tension in the string, which, for small amplitudes, takes the second derivative in position. The wave equation is written as:

$$c^2 \frac{\partial^2 \psi}{\partial t^2} = \nabla^2 \psi \tag{8}$$

It is also one of the most important equations in all of physics and in all of science. It turns up in countless forms and wide varieties of systems. Ultimately, all waves can be approximated by this equation to first order. Other terms added on such as dissipative mechanisms and nonlinearities can change the solution (we will talk about this later).

The most common solutions to waves are sinusoidal functions:

$$\psi(x,t) = \cos(kx - \omega t) + sin(kx - \omega t) \tag{9}$$

where k is called the wave vector which is defined as

$$k = \frac{2\pi}{\lambda} \tag{10}$$

where λ is the wavelength of the wave. The wave equation is so popular because of its sinusoidal solutions. This solution in particular considers waves traveling in the positive x direction. Sinusoidal functions describe structures that repeat over and over in either time and space.

Knowing both the frequency and the wave vector of the solution to the wave equation fully defines a wave, its period, its wavelength and we will soon see that these can define the velocity(s) of the wave. If we write out ω/k, we are in units of position per time which is also the units for velocity.

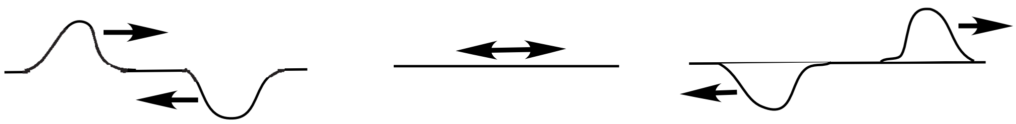

Figure 5: This is deconstructive interference. One can imagine the opposite would occur if the waves were both positive.

In considering this solution for most cases, a wave transports energy and momentum without transferring mass. Waves propagate through a medium, whether it be ocean waves through the water, sound waves through the air, light waves through an electromagnetic field, sports waves through crowds of people, demographic waves, traffic waves etc. Commonly waves have a varying amplitude through time and space dictated by their wave speed.

A special phenomenon with waves in the most basic sense is that they also add linearly. A schematic is shown in Figure 4. When one wave of a fixed wavelength encounters another wave traveling at a different speed or direction, the resulting amplitude is determined by linearly adding the amplitudes. The result is both constructive and destructive interference. For example, consider two waves approaching each other from opposite directions. As they reach the same point their amplitudes linearly add and for that brief moment you have twice the amplitude. If one wave is the crest and the other wave is a trough, as they pass through each other, they will temporarily dissappear before emerging out on the other side.

There are some common examples of constructive and destructive interference. One example is at a concert when there are two speakers emitting a sound. The two "wavemakers" create a map of constructive and destructive interference patters depending on how the waves add in a particular region. Depending on where you are, you may hear resonating, amplified sound or nearly no sound at all. Another example is the ideas of sets and lulls in the ocean. As waves are constantly traveling from different directions and at different speeds, constructive and destructive interference is likely to occur over the course of several waves[1]. This manifests itself as periods of larger swells (sets) followed by periods of smaller waves or no waves at all.

2.3 The Dispersion Relation: Wave Speed

A glass prism that splits white light into the rainbow is a great result from the dispersive nature of waves. This occurs because the different frequencies of light have the same speed so they travel at different angles through an air glass interface. As the light comes out of the glass it is clearly separated into all of the colors. Similar things occur in ocean waves which we will later see.

We have already defined wave vector and frequency of a wave. Now we are able to define two velocities in a wave. The group velocity is:

$$v_g = \frac{\omega}{k} \tag{11}$$

while the "phase" velocity is

$$v_p = \frac{\partial \omega}{\partial k} \tag{12}$$

The group velocity describes how fast the perturbance moves or how fast the energy travels in the medium. The phase velocity on the other hand describes how a particular region or shape within the perturbation travels. Sometimes they are the same thing and sometimes they are not! It all depends on the medium they are travelling in.

Dispersive propagation occurs when the phase velocity depends on frequency. Most of our acquaintance with waves is with nondispersive propagation, in which the phase velocity is independent of the frequency. In that case, the phase velocity, the energy velocity, and the group velocity are all equal to each other and constants. In general, this simple relation does not hold.

Surface waves on water are one common example of dispersive wave propagation that is easily observable. If the depth h of the water is much less than the wavelength, then the phase velocity is only dependent of the depth and is independent of wavelength and the propagation is nondispersive.

On the other hand, if the depth of the water is much greater than the wavelength, the phase velocity is

$$v_p = \frac{g}{k}. \tag{13}$$

The group velocity v_g is defined as the derivative $d\omega/dk$, and is about the speed at which a localized disturbance will travel, and is often the speed at which energy travels, as it is in the case of these waves. The group velocity of deep-water waves is, therefore,

$$v_g = \frac{v_p}{2} \tag{14}$$

If you drop a pebble in still water, a circular ring of disturbance moves out at the group velocity. If you look closely at the ring, you will see that waves are being created at the inner boundary of the disturbance that move outwards faster than the disturbance itself, and disappear at the forward edge. This is an excellent demonstration of the difference between phase and group velocities. It is also clear that the energy moves at the group velocity. For this to succeed, the water must be considerably deeper than the wavelength of the waves in the disturbance. In shallow water, the disturbance and the wavelets will move in unison.

We will find in future sections that waves in a "swell's" group velocity is wavevector (wavelength) dependent. Therefore, a localized swell will spread itself out with the faster waves of longer wavelength arriving at the coast first and could be followed by slower waves possibly several days later. We will also see in the upcoming chapters how the dispersion relation can be counteracted by "nonlinearity" and "localize" a wave dispersion that would normally spread itself out.

2.4 Boundary Conditions

Wave solutions are specified with the use of boundary conditions. Boundary conditions are information about the solution at certain places. Let us consider a string of length L which we would like to solve for the wave solutions upon it. The boundary conditions are such that

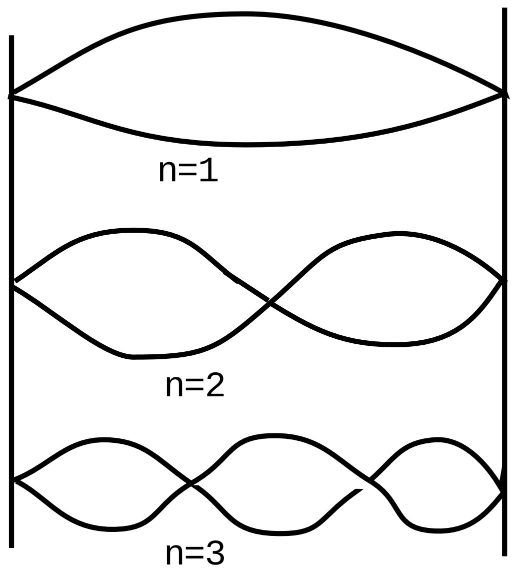

Figure 6: These are the solutions (higher harmonics) to the wave equation with boundary conditions imposed.

$$\psi(x = 0, t) = 0 = \psi(x = L, t) \tag{15}$$

Then we know that solutions can only be of the form:

$$\psi(t) = A\sin(\frac{2\pi n}{L}x) \tag{16}$$

for n=1,2... in order to satisfy the boundary conditions.

Other boundary conditions include having an "antinode" at a boundary which is the case in some musical instruments like trumpets and saxophones

A little later on in Chapter 7 we will see how ocean waves interact with the coastlines including island and how the boundary conditions dictate their general behavior such as reflection, transmission and scattering.

2.5 Fourier Analysis

Fourier analysis will be the last stop on our review of wave mechanics. The power and the beauty of Fourier analysis will prove to be very important in any study of waves and many other fields of physics and mathematics. The main idea of Fourier analysis is as follows:

A collection of waves can be decomposed into their spectral densities.

Joseph Fourier developed his ideas on the convergence of trigonometric series while studying heat flow. Fourier neither invented "Fourier" series nor settled any of the outstanding question, but he did use it fruitfully especially for problems regarding the conduction of heat in solids. His paper in 1807 was rejected by other scientists as too imprecise and was not published until 1822. His ideas come down to the fact that the trigonometric functions can be used as a "basis set" which, with the proper coeffeicients, can construct any function. In this context, any shape, any construction, anything can be built out of or decomposed into its Fourier components.

Fourier Series can be imagined as expanding a given vector in terms of a set of orthogonal basis vectors (x, y, z...) The Fourier Transform of a function f(x) is defined as

$$F(\omega) = \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx \tag{17}$$

Let us consider an example of the ocean surface that looks as is shown in Figure 6. The Fourier transform of this surface would simply be shown in Figure 7.

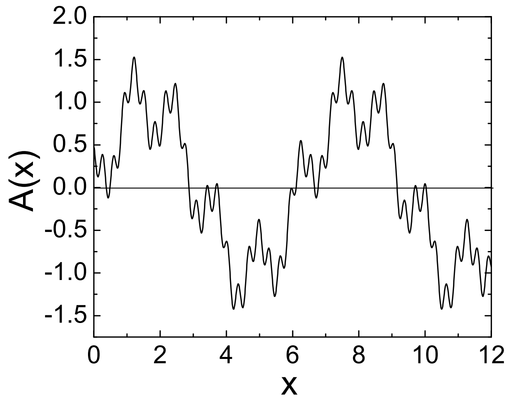

Figure 7: This is a function containing three different sine waves linearly added together.

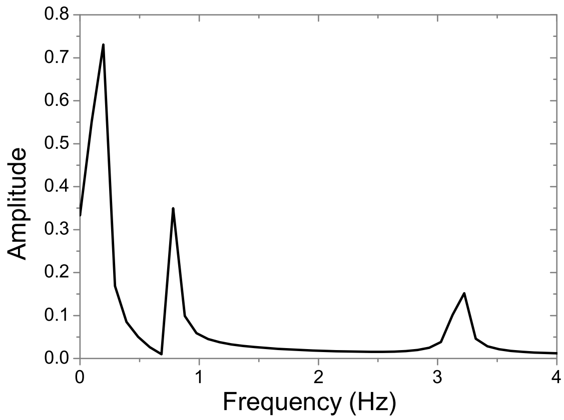

Figure 8: This is the Fourier Transform of the the previous function.

3 Fluid Dynamics

'Sea" is the name given to that water which is wide and deep in which the waters have not much motion.

-Leonardo DaVinci (Notebooks Vol 2)

Fluid behavior is a fascinating physical phenomena. Although the subject is neglected in most scientific training except possibly for a mechanical engineer, fluid dynamics is an extremely important subject for things from airplane flight and boat transportation to biological processes on the microscopic scales. The difficulty in which such dynamics can be described has troubled scientists for centuries. We will soon see why fluid dynamics is inherently such a difficult subject to describe from a fundamental physical standpoint. Because of this inherent complexity of fluids, we can only scratch the surface of the field. For more substantive investigation, the reader is directed to more thorough accounts. The following discussion only hopes to give a flavor for the main tools in fluid dynamics, some of the descriptions and some of the problems or shortcomings with the field that are still presently under investigation.

3.1 Conservation Equations

A few general assumptions are all that we need to start talking about fluids. These are namely the conservation of mass and the conservation of density.

The equation of continuity is a very useful relationship derived in many fields in physics including electromagnetism and thermodynamics. Fluids are no exception.

The equation is simply based on the idea that some things cannot be created or destroyed. If we consider a region in a fluid, we know that the rate at which the total amount of fluid entering and leaving that region is related to how much that region is changing in amount. So to begin, we can write:

$$\frac{\partial}{\partial t}\int_{V_o} \rho dV = \int_S \rho v dS \tag{18}$$

We can use Gauss's Law (an extension of the fundamental theorem of Calculus)

$$\int_S \rho v dS = \int_V \nabla \cdot (\rho v) dV \tag{19}$$

Combining these two equations together and throwing out the integral leaves:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) = 0 \tag{20}$$

This is the equation of continuity!

3.1.1 Fluids

Fluids include both liquids and gases. They are present in so many areas of our daily lives. Common properties of fluids include their density, pressure and viscosity.

Archimedes principle describes the way that things float. It states that a body wholly or partially immersed in a fluid will be buoyed up by a force equal to the weight of the fluid that the body displaces.

Bernoulli's principle dictates how the normal pressure due to a fluid is related to the velocity that it moves. It says "If the speed of a fluid particle increases as it travels along a streamline, the pressure of a fluid must decrease and vice versa. Many of us are familiar with Bernouli's equation for fluid flow.

$$\rho + \frac{1}{2}\rho v^2 + \rho g y = constant \tag{21}$$

This is essentially applying energy conservation to a continuum fluid system[2].

3.1.2 The Euler Equation

The most essential part of fluid dynamics is best thought of as a field theory. Similar to electromagnetism, in which, say, an electric field is sought for all of space, problems in fluid dynamics seek the velocity field. From the velocity field, one can then determine forces on boundaries, energy fluxes, etc. We rely on calculus to take the limit of a continuous function as our starting point for describing the value of velocity fields and their derivatives at a specific point. This is the basis for continuum mechanics that fluid dynamics is one very important subset of.

In fluid dynamics we consider an infinitesimal volume element of the fluid as shown in Figure 8. The net force on a volume element is the change in pressure across that element (as well as other forces). We can write this as (just as in the case for a harmonic oscillator in which we compared two definitions of what the force is)

$$m\frac{d^2x}{dt^2} = -V\frac{\partial P}{\partial x} \tag{22}$$

which can be written in three dimensions as

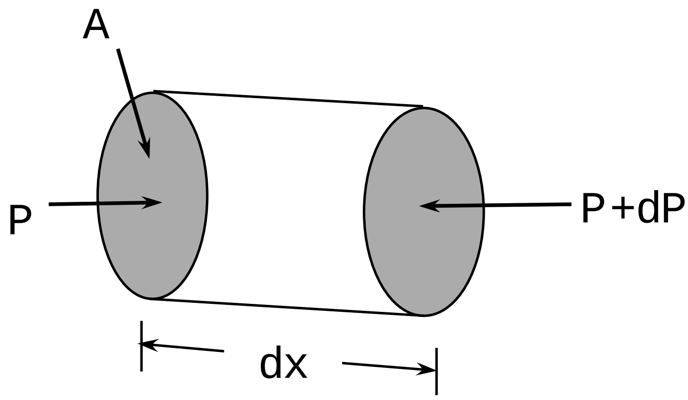

Figure 9: This is a fluid element. The difference in pressure and other forces dictate how the velocity at that point changes

$$\rho\frac{Dv}{Dt} = -\nabla P \tag{23}$$

As a brief detour, if we want to talk, in general, how a function, f, depends on t(time) and also x (position) which is itself a function of time then we have to write:

$$\frac{df}{dt} = \sum \frac{\partial f}{\partial x_i}\frac{\partial x_i}{\partial t} + \frac{\partial f}{\partial t} \tag{24}$$

but dx/dt by definition is v, so we can write (generalized to three equations)

$$\frac{df}{dt} = v \cdot \nabla f + \frac{\partial f}{\partial t} \tag{25}$$

So if we use this relationship for velocity, and consider how it changes with time (its acceleration) and then we go a little further and apply Newton's second law to a fluid "element", its equation of motion becomes

$$\frac{\partial v}{\partial t} + (v \cdot \nabla)v = -\frac{1}{\rho}(\nabla P + F_{ext}) \tag{26}$$

This equation happens to be the most important equation for describing a majority of phenomenon found in the ocean. This is the Euler equation. It is important to emphasize that the fluid is incompressible so ρ is constant. Later on we will consider cases where it varies with position or temperature making things alot more difficult! The left hand side of the equation is called the "convective derivative". The convective derivative is inherently nonlinear because its "operator" contains within it the function itself. Nonlinear equations are even more difficult to solve than linear pde's such as the wave equation. Nonlinear equations cannot generally be described analytically and therefore must be considered in a more qualitative and limit taking case. In many cases, ways are sought to simplify the equation and neglect the nonlinear term.

It is useful to note the relationship:

$$(v \cdot \nabla)v = 1/2\nabla v^2 - v \times (\nabla \times v) \tag{27}$$

for problems we may run into in the future. In many cases, fluids are assumed incompresible and irrotational. This simplifies the math alot as we will later see.. [3]

3.2 Navier Stokes

Although the Euler equation is very useful for describing many fluid processes (in the ocean for example), there are cases in which the validity of the solution breaks down. The main reason the Euler equation is not the end of the story is partly because the effect of boundaries. When we consider boundaries, we realize that if a fluid has a viscosity, not only must the normal component of the velocity be zero, but the tangential part as well. Viscosity is what we use to explain why fluids cannot "slip" at the boundary. The viscosity of water is $0.01cm^2/sec$ for water. Viscosity is simply defined as the resistance to flow. In the case where viscosity becomes important the Euler equation is not exact. We need to add an additional forcing term from the viscous force. Therefore we write Newton's law as

$$\rho\frac{\partial v}{\partial t} + (v \cdot \nabla)v = -\nabla P + \gamma\nabla^2 v \tag{28}$$

where γ is the viscosity of the fluid. This equation is the Navier Stokes Equation for incompressible fluids. It is the most general and powerful equation to describe the motion of any fluid. The equation is particularly important for flow near boundaries such as through pipes or porous media. In other cases, such as large ocean waves, viscosity can be neglected and we can return back to the Euler equation to describe most of the field of fluid flow.

Solutions to the Navier Stoke's equation provide the velocity field in a region of interest. Once we solve the Navier Stoke's equation or the simplified Euler equation we have the equation of motion for how the system behaves. When the system has

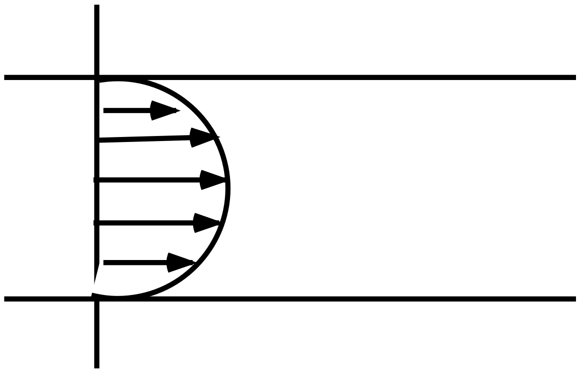

Figure 10: The flow field through a cylindrical pipe

reached steady state, the velocity as a function of position stops changing and we are left with an exact, time independent solution for what the flow is doing.

3.3 Example: Flow through a Pipe

It is useful to work out the solutions to flow through or around different geometries such as flow past a cylinder, sphere or cone. In these situations, one can exploit the angular symmetry or translational invariance. In this section, we will briefly review the solution for viscous flow through a pipe.

We consider flow through a pipe with a circular cross section whose walls are perfectly rigid. We will also suppose that the system is in steady state and that the velocity of the fluid everywhere is in the z direction. We can write down the zth component to the Navier Stoke's equation.

$$\rho\frac{\partial v_z}{\partial t} + \rho v_z\frac{\partial v_z}{\partial z} = \frac{\partial P}{\partial z} + \eta\nabla^2 v_z \tag{29}$$

Due to our conditions of steady state and translational invariance, the terms on the left hand side vanish and we are left with

$$\frac{1}{\eta}\frac{\partial P}{\partial z} = \nabla^2 v_z \tag{30}$$

while the r component of the equation yields

$$\frac{\partial P}{\partial r} = 0 \tag{31}$$

Each plane perpendicular to the z axis is at constant pressure so we can write

$$\frac{\partial P}{\partial z} = -\frac{\triangle P}{\triangle l} \tag{32}$$

The previous equation now becomes:

$$\frac{1}{r}\frac{d}{dr}(r\frac{dv_z}{dr}) = -\frac{\triangle P}{\eta\triangle l} \tag{33}$$

which can be integrated to give

$$v_z = (-\frac{\triangle P}{4\eta\triangle l})r^2 + C_1\ln(r) + C_2 \tag{34}$$

As with any differential equation, one must impose boundary conditions such as the flow being finite everywhere including r=0. This makes C_2 become zero. If we consider for a moment, the flow at the outer boundary (as fluid atoms are colliding with the rough wall) we realize that v_z must go to zero on the the outer barrier as well! Therefore the solution to the velocity is simply

$$v_z = \frac{\triangle P}{4\eta\triangle l}(R^2 - r^2) \tag{35}$$

This flow situation is commonly referred to as Poiseulle flow[3].

3.4 The Velocity Potential

Having derived the Euler equation and writing out the full Navier Stoke's equation, we can now make some very general statements about the type of fluid motion that can occur. If the velocity field is such that:

$$\nabla \cdot v = 0 \tag{36}$$

the fluid density does not change and only motions which conserve density are allowed. If the field is such that

$$\nabla \times v = 0 \tag{37}$$

then no rotational motion is allowed in the fluid. For obvious reasons, such a flow is called irrotational flow. If we have irrotational flow, then the velocity can be written as

$$v = \nabla\phi \tag{38}$$

where ϕ is a scalar function called the velocity potential. Thus, irrotational flow is sometimes called potential flow. Many examples in fluid dynamics involve potential flow which is fortunate since the introduction of a velocity potential allows us to work with scalar rather than vector quantities.

In the special case of potential flow of an incompressible fluid we have:

$$\nabla^2 \phi = 0 \tag{39}$$

which is just Laplace's equation (which is rather fun to solve). It is a popular partial differential equation and has some nice characteristics like linearity. We will explore it later, particularly in the problem of finding solutions to surface ocean waves. We can write down Euler's equation in terms of the velocity potential. After fiddling around a little from using equation 27, we are left with:

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}v^2 + \frac{P}{\rho} + F = q(t) \tag{40}$$

where q(t) is an arbitrary function of time and plays the role of an integration constant. If in addition to being irrotational, the flow has achieved steady state, then the time derivative vanishes and we are left with:

$$\frac{1}{2}v^2 + \frac{P}{\rho} + F = constant \tag{41}$$

This linear equation is a convenient equation for the description of some fluid systems and will be used later on in the text.

3.5 Reynold's Number

We all know intuitively, that the fluid dynamics of different systems behave rather differently. Stirring around a cup of honey is rather different than the dynamics of rising smoke. Many of the difference between fluids can be accounted for by the Reynold's number. The Reynold's number defines the relative importance of viscosity as compared to inertial effects.

$$R_e = \frac{\rho V L}{\eta} \tag{42}$$

where V is the typical velocity and L is the "typical length". For high reynold's numbers, the inertial term dominates, the viscous term can be dropped and we are left again with the Euler equation. On the other hand, low reynold's numbers are dominated by the viscous term and dampening and dissipation becomes important. For macroscopic fluids such as the ocean, the Reynold's number is huge, many 1000s so we can neglect viscosity. On the smaller scales such as smaller fish swimming or capillary waves, the Reynold's number may become low enough that viscosity is significant.

The Reynold's number provides an interesting result. If two systems with different viscosities and different sizes etc have the same Reynold's number then we are left with the same solution to the flow. This is called the law of similarity and shows that fluid flow is ultimately scale independent when the correct parameters are considered

4 High Reynold's Number Fluids

4.1 Instabilities

Let us not forget the reason for embarking on this task of understanding fluids: to talk about the dynamics of waves and wave generation. This relates to the fields of meteorology, oceanography and geology.

A glass of water sitting on the table is probably the least interesting problem in all of fluid dynamics. Fluids that flow are alot more interesting. Fluids only flow when they are being driven by an external force, whether it be something like gravity, pressure gradients or wind. The dynamics of driven fluids are a fascinating area that is currently being extensively investigated.

There are two main directions to go from the standard dynamics of driven fluids. One direction to go will be at low reynold's number and even near the liquid solid transition. This regime may be called complex fluids. The idea here is that the system is so viscous it is almost a solid. An example of this is the cooling of glass. At high temperatures, the material flows around alot like honey and enables glass makers to form it into all sorts of neat things. But as it cools it becomes more viscous. When exactly does it become a solid? Nobody really knows. This field of complex fluids helps to explain many geological processes in terms of solids flowing and cracking, sand dunes forming, erosion and other interesting phenomena in geology.

The other direction to go in is the direction of high reynold's number, low viscosity and rapid flows. This is the arena of chaos, turbulence and the dynamics of dilute gases. This is the area of meteorology and many dynamics of the ocean, convection and other unstable flows that we will briefly discuss. In this regime, the idea of equilibrium and instabilities is very important. For a system to be in stable equilibrium, not only must we have a situation in which all forces are in balance, but where small deviations of the system from equilibrium must generate forces which tend to drive the system back towards its equilibrium configuration instead of farther away from it. A ball sitting on top of a hill would be an example of unstable equilibrium.

Linear instabilities cause the growth of fourier modes (wavelike solutions to the nonlinear equation that are stable with respect to perturbations). This makes the field of instability analysis synonymous with "pattern formation". It is common for only a few patterns (modes) to be stable and pattern like solutions to abound in nature.

4.1.1 Convection

When a fluid is heated, the heat can be transferred throughout the material by conduction or convection. Convection is an example of an instability that occurs in a fluid for large enough temperature gradients. Particularly in the presence of a gravitational field, when a fluid is heated from below, it is easy to imagine for the warm fluid will rise and the cold fluid will fall. This is experimentally verifiable by boiling a pot of water and watching the convection currents in the system (it is easier

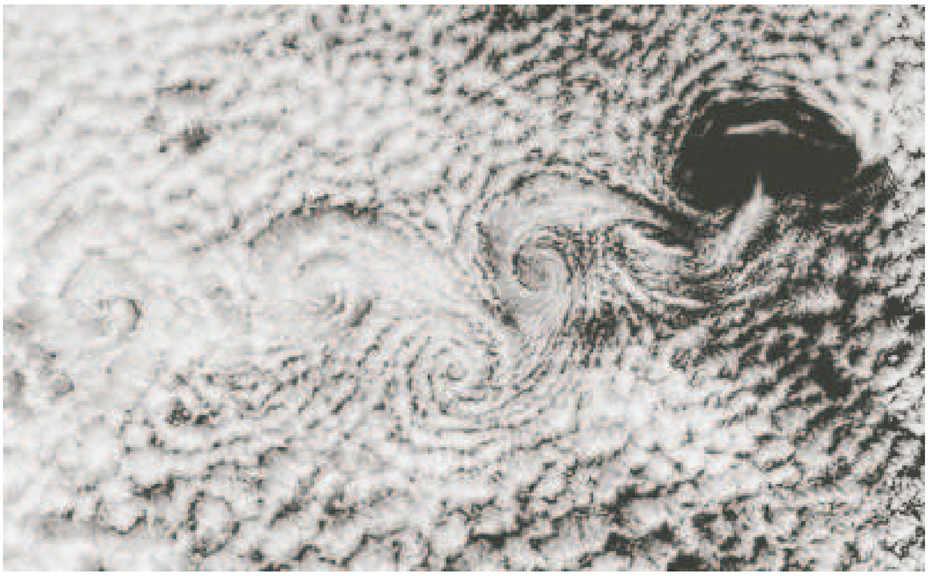

Figure 11: We will consider one important example for the onset of turbulence: flow past a cylinder. For increasing the Reynold's number in flow past a cylinder we break the following symmetries in order: 1)left-right asymmetry R=1.54 2)Vertical symmetry (Andronov Hopf) bifurcation which makes the flow time periodic (breaking time invariance) giving Karman vortex street. Finally, Lagrangian turbulence sets in at about R=240 in which chaotic advection occurs. It is this limit that turbulence becomes isotropic. For even higher speeds, we can observe the onset of turbulece. These critical velocities for the system are examples of onsets and critical behavior. Such critical behaior is very interesting. This is a simple example that can be used to describe alot of interesting systems in fluid dynamics such as the weather patterns or ocean currents.

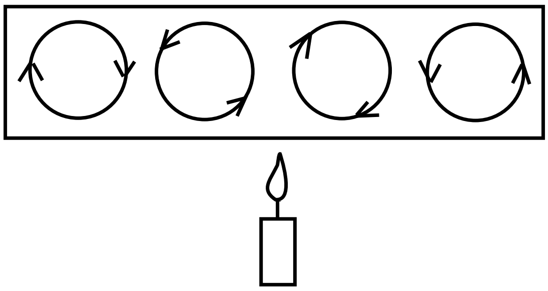

Figure 12: The general setup for a convection experiment.

to do if you are boiling rice and you can more easily see the streamlines). Another example is a cup of hot Miso soup in which convection rolls are easy to note. These types of convecting systems under the presence of gravity are found everywhere in nature. The patterns found in one's Miso soup are surprisingly similar to the convection rolls found on the surface of the sun! Figure 10 shows an example of convection rolls in an experimental setup.

To describe the dynamics of a convecting system we have to partially change the initial assumptions about the incompressibility of a fluid. For many fluid systems, the assumption of incompressibility was rather good. For a fluid in which thermal effects are important, there is a link between density and temperature. When there is a temperature gradient there becomes a density gradient which therefore makes a pressure field. Since the hot fluid expands, the vertical temeprature gradient across the fluid results in a density gradient. As the case for an ideal gas:

$$P = \rho R T \tag{43}$$

Therefore we encounter a destabilizing buoyancy force since the colder heavier fluid would like to fall down so as to minimize gravitational energy. The viscosity of the fluid has a stabilizing effect and for small temperature gradients the fluid remains at rest and there is only heat conduction in the system. When the temperature gradient becomes large or viscosity is lowered, the unstabilizing temperature gradient dominates. The ratio of the viscosity to the temperature gradient is the control parameter of the system and determines the dynamics. This is the classical Rayleigh Bernard instability. Such phenomenon are very common for weather systems in which temperature gradients that lead to convection currents are the cause for such things as the formation of clouds and winds.

We may define α as the coefficient of expansion for a fluid

$$\alpha = \frac{\triangle\rho}{\rho_o(T - T_o)} \tag{44}$$

When temperature effects become important, we must include the heat equation along with the Euler equation, continuity and equation of state as the basic equations that must be solved in describing the motion of a fluid.

We can write down the full (Navier Stokes) equation for a system in which there is a thermal gradient

$$\rho\frac{\partial v}{\partial t} + (v \cdot \nabla)v = -\nabla P + \eta\nabla^2 v + (g - g\alpha\theta) \tag{45}$$

Bossinesque proposed that we approximate all fluids as not being affected by varying density except in the case of density being affected by gravity. In other words, we consider the density is constant except when it is next to the gravitational force. This brings us to the field of stability analysis. Stability analysis is a giant field that we do not have the time or space to go into detail about. Stability analysis is very important for driven fluids at high Reynold's number. The main process for stability analysis is as follows: Apply a small perturbation to a known solution to the equation. Then check to see whether the perturbation grows or decays away. We can go on to analyze the stability of this function in relation to the control parameter. Using stability analysis and determining the response of the system to a small perturbation, we can see whether the dynamics decay away or exponentially increase. In general this equation is in equilibrium for small temperature gradients. For large ones, the solution becomes unstable and perturbations grow instead of die away. Such new solutions lead to wave solutions and circulatory type behavior. In a standard heat conducting cell this would lead to stripes when viewed from the top or circular motion when view from the side.

This is similar to what happens in both the ocean that lead to ocean currents and the atmosphere that lead to weather patterns, low and high pressure systems.

4.1.2 Scaling and Power Laws

In the following sections we will be introduced to various phenomenon that have a interesting property which we can call "symmetry of scales". This means that if you look at something and note its general characteristics, shape of structures, motion of features, etc and then zoom in an order of magnitude or more closer, you may find that the system looks almost exactly the same. This is scale symmetric and is

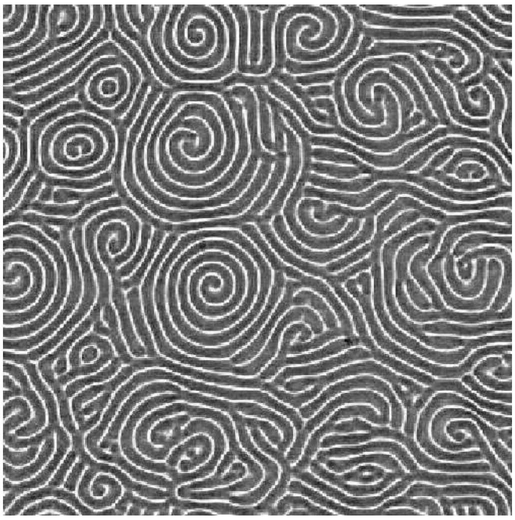

Figure 13: A photo of convective pattern formation due to heating of the bottom plate and looking from above. Similar patterns are formed in both our atmosphere and our oceans. (Courtesy of Ahlers (UCSB))

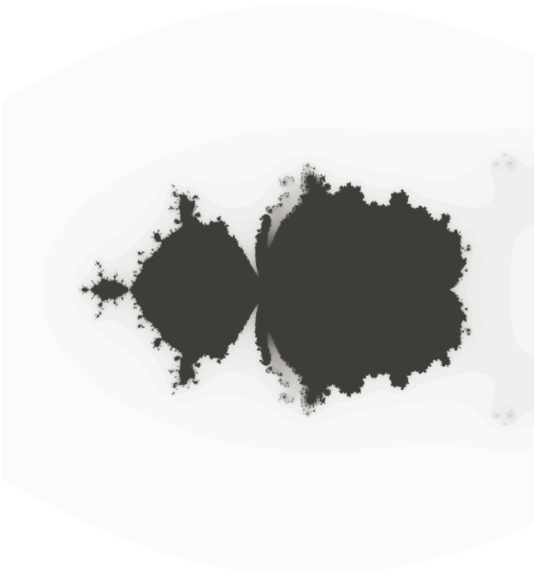

Figure 14: A fractal. If you zoom in on it, the tiny structures wil look alot like the biggest structure.

related to the term fractal. A simple example of a scale symmetric object is a tree in which a tree branch looks very much like a smaller tree.

Nature and earth systems abound with fractals and systems that scale. Scaling in the ocean is very important for understanding some situations. Turbulent waters will have small waves on top of larger waves on top of giant waves. Referring to Figures 6 and 7, there would be some proportionality factor between the Fourier peaks that could quantify the scaling. Ocean currents or geological structures, coastlines and ocean bottoms can have fractal nature as well. In the next section we will briefly explore one example of scaling; the mystery of turbulence.

4.1.3 Turbulence

In considering the case of Rayleigh Bernard convection, solutions to the Navier Stokes equation quickly become very difficult to solve when terms can no longer be neglected and simple assumptions such as as incompressibility no longer hold. Strong nonlinearities nested in the equation are the main source for this difficulty. In these cases, particularly for fully developed turbulence, statistical descriptions become much more useful in describing fluid flow.

Every beach goer is familiar with the choppy conditions associated with high winds and every southern californian surfer is familiar with eddy conditions (not many people like eddy). These phenomena are due to instabilities in the Navier Stoke's equation. How boring the world would be if the Navier Stoke's equation were linear and convection of the atmosphere never occurred and the seas were always glassy! Turbulence is the classic example of fully developed chaos and instability. As in the case of convection, after a particular control parameter surpasses a critical value, a marked bifurcation occurs indicated by a significantly different appearance of the dynamical flow behavior. One has seen many times a glassy ocean with a light breeze on it. As the breeze continues to slowly increase, the ocean seems to abruptly change to choppy.

The Navier Stokes Equations probably contains all of turbulence. However, going through the steps that may lead to reasonable descriptions of the behavior has proven to be an insurmountable task. The degree of turbulence in a system is determined by the Reynold's number that we have discussed before. Due to the similarity principle for incompressible flow, for a given geometrical shape of the boundaries, the Reynold's number is the only control parameter.

Besides experimental verification and instability analysis, there has some some progress in quantifying the dynamics of turbulent systems. These are some of the statistical descriptions of fluid flow. Such descriptions are in sharp contrast to the nature of solving PDE's. This primarily includes two empirical laws of fully developed turbulence.

1)In turbulent flow at very high Reynold's number, the mean square velocity increment $< v^2 >$ between two points seperated by a distance l behaves approximately as the two thirds power of the distance.

2)Kolmogorov's four fifth's law: In the limit of infinite Reynold's number, the third order longitudinal structure function of homogeneous isotropic turbulence eval-

Figure 15: Turbulence in rising smoke due to high Reynold's number.

uated for increments small compared to the integral scale, is given in terms of the mean energy per unit mass ϵ by

$$< (\partial v_{\parallel}(r,l)^3 >= -4/5\epsilon l \tag{46}$$

Another useful way to consider turbulence is with the idea of self similarity or symmetry on scales. In other words, structures on a large scale (100s of kilometers long) look and behave exactly the way that structures on smaller scales do. This is another example of the term fractal. Fractals appear in nature in many forms other that turbulence such as the shape of a coastline, or the limbs of a tree[4].

4.2 Applications to Meteorology

If one takes a step back and looks at the entire atmosphere of the planet, the fluid dynamics happen to be surprisingly difficult. The fluid is compressible and rotational. It consists of many materials and many phases such as air and condensed water vapor. It has very irregular boundary conditions on the mountainous land and oceans. On top of this, the dynamics all occur on the spinning earth, a noninertial system[5].

What, then can be said about the dynamics of the atmosphere? In very general terms, scaling is an important feature of the atmosphere. This include the general circulation and currents, Rossby waves, hurricanes, squalls, tornadoes and even dust devils. Although we may not understand fully the dynamics on any one of these scales, it is comforting that the dynamics on these different scales is similar. Rossby waves are well defined dynamical systems that generally move from east to west in the middle and high lattitudes. In the upper part of the atmosphere, they consist of smooth wave shaped patterns with cold troughs and warm ridges. Typically one finds about five cycles of Rossby waves wrapped around the hemisphere [6].

Convection happens in many instances in the atmosphere. Updrafts occur near cities where the air is warmer than over the countryside. They can happen on much smaller scales such as paved parking lots or even islands within the cooler ocean.

When considering the earth, the most simple observation is that it is warmer at the equator than on the poles which would lead to two simple convective roles, one for each hemisphere.

We can propose a convective model of the atmosphere based on the idea of Bernard cells assuming that there is uneven heating on the earth between the equator and the poles. A slightly more specified model of the earth's atmosphere include three cells for each hemisphere, from 0 to 30 the wind is easterly, from 30 to 60 it is westerly (including most of the temperate zones of the United States) and then is easterly again for the most polar lattitudes.

Using similar arguments as for the parallel plate setup, we can accurately describe the currents of the atmosphere to first order. We get strong dynamical consequences of the motion of the ocean and the atmosphere by the rotation of the earth. We know that if a force acts in an inertial system in such a way as to produce an acceleration a_o then that same force acting in a rotation coordinate system will produce an apparent acceleration given by:

$$a = a_o - 2\omega \times v - \omega \times (\omega \times r) - \frac{d\omega}{dt} \times r \tag{47}$$

where ω is the frequency of rotation of the coordinate system. If we are sitting in a coordinate system fixed on the surface of the earth, then we can take $d\omega/dt = 0$. The two extra terms are then the familiar centrifugal and Coriolis forces.

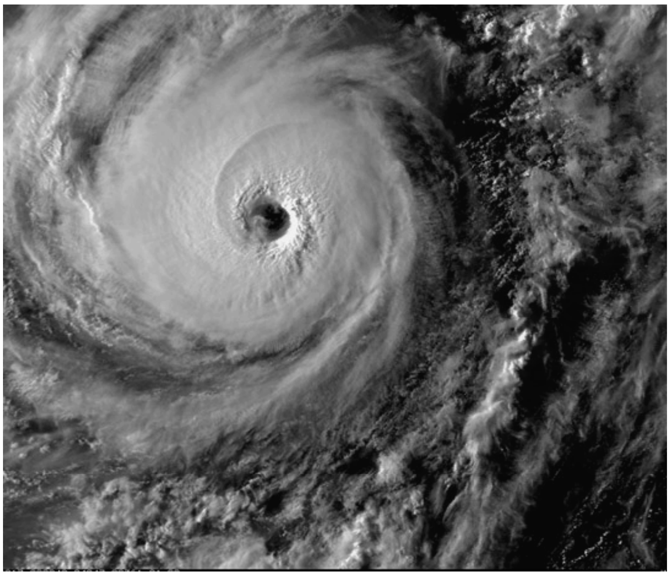

Figure 16: The first hurricane of the 2004 Atlantic Season. An intense area of low pressure creates convective forces and stron winds. The spinning of the hurricane is caused by the Corriolis force and the spinning of the earth.

5 Geology and Other Complex Fluids

5.1 Complex Fluids

Geology is approached oddly from a physicists perspective. This section will not include types of rocks or geological time periods or any of the things most commonly associated with geology. Instead, we will use the approach from the last few chapters on fluid dynamics to approach this subject. Physicists can assume that geological systems are just another fluid system where fluctuations and small variations in parameters account for the rich variety of structures that we see.

We have seen some of the interesting behavior of fluids at high Reynold's number using weather as the most important example. On the other side of the coin from the fluctuating, nonlinear behavior of high reynold's number fluids are complex fluids (low reynold's numbers). Complex fluids, in the most general sense, are always close to the division between a solid and a liquid. Shaving cream, a collection of gaseous bubbles separated by liquid walls is a great example of this type of material. Although a pile of shaving cream may sit as though it behaves as a solid, smearing it with a little applied force shows its ability to flow and behave like a liquid. Geological systems are also excellent examples of complex fluids. The earth is a system appearing static and timeless. However, dynamic processes such as erosion or earthquakes show these geological system's ability to significantly change over time. Ultimately, the rocky crust of the earth that we sit upon is flowing (or at least cracking) at this very moment.

There are a few ways to consider the transition between high and low Reynold's number regimes. One is that lower Reynold's numbers are at higher densities, shorter length scales, lower driving forces or closer to the transition to solids. The disordered particles that make up a complex fluid contribute to the irregular nature of the flow. Its interesting to consider that fluids can have solid properties and solids can have fluid properties. Generally speaking, the main indication of complex fluid flow is long time scales. The "Deborah number" which describes the timescales needed for a system to relax is useful in considering such fluid systems. This number is associated with the Biblical prophet Deborah who said "The mountains flow according to the Lord's time scale and not humans." In this sense, while "normal" fluid time scales are less than a second, the Deborah number in complex fluids and geological structures can be millions of years! Due to the slower nature of these systems, sharper time resolution is available to study the dynamics. Using earthquake records for example, due to this sharper time resolution, more random and fluctuating behavior can easily be noted. It all depends on the timescale we are using to talk about how it behaves.

It is within the context of geology that we find complex fluids most easily understandable. In considering the unusual and "impossible to predict" fluctuating nature of earthquake records that we can gain insight into the nature of complex fluids. This earthquake data contains alot of very small fluctuations in the motion of the plates at irregular times with an occasional large earthquake every so many years or centuries, depending on how unlucky you are. If we plot the probability that a certain size earthquake occurs, we often get a power law distribution. This

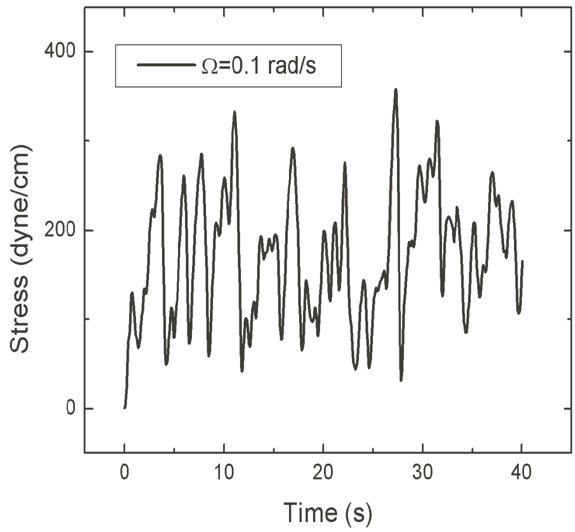

Figure 17: Stress versus time plot for a foam system under steady shear. Such data has striking similarities to earthquake data.

is often referred to as self organized criticality since the size and distribution of earthquakes are independent of the time scale you are measuring at.

Simple views of solids and fluids use understanding of the microscopic properties to determine macroscopic behavior and introduce continuum models. Complex fluids have important characteristics on the mesoscopic scale or multiple length scales. For example, when considering a geological structure its obvious that we have structures ranging from fine sand, coarse sand, pebbles, small rocks, large rocks, boulders, mountains, tectonic plates. Likewise, geological structure is made up of many different types of rocks, sedimentary, lava rocks etc. Stresses applied to these systems and mixtures of materials allow the forces to be translated to all these different length scales and degrees of freedom. Fluctuations and small variations in density and structure are the most general way to account for the diverstiy of geological features that we observe.

These multiple length scales attribute to the macroscopic quantities. Other common examples of complex fluids include binary mixtures having a coexistence between two phases: solid-liquid (suspensions), solid-gas (granular), liquid-gas (foams) and liquid-liquid (emulsions). Such materials are often highly disordered and may have a variance in particle size. Their flow properties are influenced by the geometrical constraints resulting from this disorder. It is also uncertain what role the particle interaction or defects may have in the flow properties of these interesting materials[**?**][**?**].

I don't want to dive in too deep into complex fluids, we may never be able to get back out. We'll just go over a few brief examples of them and some of the associated phenomena just to get a taste. Otherwise, we can refer to all the other literature about complex fluids for those with a bigger appetite. Now lets look at a few table top examples.

5.1.1 Foams

Everyone has seen foamy beers or maybepulled off a floater on a frothy broken wave. Pierre Gilles de Genne devoted his noble prize in 1991 to "soft matter", of which foams are an excellent example. Foams are gas bubbles separated by liquid walls. The system attracts interest because they can be studied on many length scales. These scales include the chemistry of surfactants on the microscopic level,

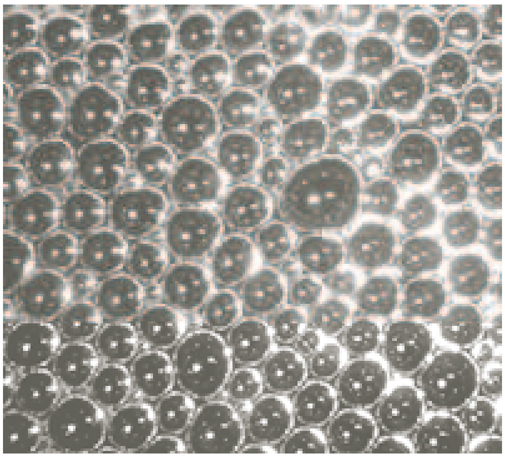

Figure 18: The interesting structure of foam.

the physics of of individual bubble topology and the engineering of macroscopic mechanical properties of large bubble systems.

5.1.2 Granular Systems

Sand is the material most likely to be found on the coastlines. Granular systems have captured much recent interest because of their rich phenomenolgy and important applications to industries such as oil drilling, agriculture and pharmaceutical companies. Many studies of granular systems driven by an external energy source have focused on energy localization, aggregation and other types of pattern formation. Other experimental work focus on the flow and mechanical properties. These experiments take place in set ups such as chute flows, horizontally oriented rotating cylindrical flows and 3D and 2D couette flows[?]. Some of these studies have focused on the stress releases that are also inherent in these systems.

As we have seen in turbulent fluids statistical descriptions are necessary for

Figure 19: The fractal like structure of mountain ranges, also well described by a random walk (Brownian Diffusion)

non deterministic systems. Granular systems are commonly another example of nondeterministic systems in need of statistical descriptions. The most successful implementation of statistical description include the theory of equilibrium thermodynamics in which the assumption that "all states are equally probable" allowed scientist to to generalize a huge variety of different systems with large number of particles. In future research, scientists may be able to find similar descriptions for these fluid flows.

5.2 Pattern Formation

As in fluids at high reynold's numbers in which patterns can be formed, the same is true for complex fluids. Often times, the fluctuations in geological structures can be quantitatively described by statistical distributions and fractal structure. Mountain ranges, coastlines and the dendritic structure of rivers are fractal like[7].

5.2.1 Sand Bar Formation

Unstable waves have long been studied in fluid shear layers. These waves affect transport in the atmosphere and oceans. Shear transmission in granular flow is

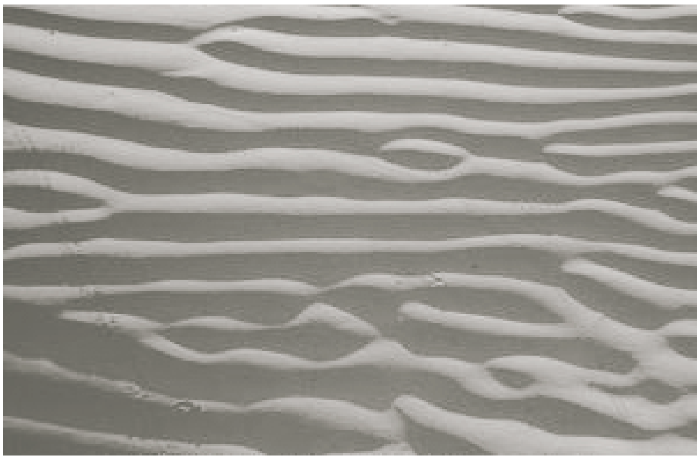

Figure 20: Granular systems driven by the flow of water or water can often exhibit some interesting patterns.

intrinsic to systems ranging from geophysical flows to industrial processin and peculiarities in granular shear response have consequences that can be dramatic of commonplace. More spectacular examples include catastrophic events such as landslides and earthquakes. Research into the shear of granular systems has attempted to clarify the mechanisms of granular repose to controlled stresses. But granular beds often display solid like and fluid like qualities side by side within a single experiment applicable rules underlying the repose of a granular be to applied shear remain elusive[8].

5.3 Surfspots

The wide variety of geological structures and erosion processes is remarkable. We have only touched the surface of what types of phenomenon can occur on these longer time scales and wide range of correlated length scales. Such diverse geological features account for a wide equally varied places where the oceans meet the land to break.

Some surfspot types include reef point breaks (Low Trestles, Salsipuedes Baja, Playa Negra Costa Rica) or Coastal shoulders (Ricon, Upper Trestles, The Cove at Sandy Hook, Malibu, Swamis, Honolua Bay) beach breaks (Huntington, Seaside Heights, Blacks Beach). Then there are also man made structures that occasionally account for very good surfing conditions including Jetti's (Manasquan, Sebastian Inlet) and Piers (Huntington, Seaside, Oceanside). We will talk about these more in Chapter 8.

6 Ocean Waves

"Now the next waves of interest, that are easily seen by everyone and which are usually used as an example in elementary courses are water waves. As we shall soon see, they are the worst possible example, because they are in no respect like sound and light; they have all the complications that waves can have.

-*The Feynman Lectures on Physics* Vol. 1 Section 51.4 (86)

Finally we arrive at the main subject of the book. We have reviewed some mathematics of the wave equation and fluid dynamics. The slower time scales of geology set the stage for incoming waves.

One of the most important aspects of fluids is the wide variety of waves which can be generated and sustained in them. The theory of water waves has been an intense scientific research subject since the days of Airy in 1843. Stokes spent a great deal of effort studying the ocean and came up with a solution in 1847 to waves striking a surface. Surface waves can be considered as simply an interface deviation between two fluids (air and sea). We will begin by discussing approximate methods to understanding of ocean wave dynamics. In actuality, real water waves propagate in a viscous ocean over an irregular bottom. They grow from external forces and internal instabilities and decay due to friction and diffusion. Approximate and simplified models of wave propagation are surprisingly helpful in understanding the most fundamental features of ocean waves.

We will first ask the question: Where do waves come from? Two components lead to waves: turbulence and Fourier series. Turbulence is due to strong winds that create disorder and inhomogeneities on the ocean surface. Waves most commonly are formed from fetch: a region where the wind is blowing in a prominent direction for some region of space. The shapes that form, we can Fourier transform it to see the components that build it up as shown in Figure 16. The greater the area of wind and the stronger the wind blows, the greater the fetch and the greater the resultant waves. Emerging from a fetch is a collection of waves known as the wave train. For reasons that will become obvious later, waves of the longest period will extend out first over the wave train while the waves of smaller period which travel slower will be at the back.

The spectral density of ocean waves that exist in our ocean span over a wide range. A spectrum would include peaks at the tidal periods of 24 and 12 hours. At the lower end there is some structure for different capillary waves. Beyond these extremes there is a broad hump spanning a period between 30 seconds and 1 second with a peak at around 10 seconds. This is exactly the period of waves that are most easily rideable by humans. Maybe god is a surfer after all!

The waves surfers are interested in are considered the long wavelength limit. Any surfer know that a rideable wave corresponds to a "wave period" of at least 10 seconds. Likewise, from watching waves break from above, on a cliff or a pier, it is easy to guess that waves are spaced out by about 100 ft as an order of magnitude guess. Thus, knowing the characteristic frequency of an ocean wave (0.1 seconds) and a characteristic wavelength (100 m) we are able to guess a wave velocity of about 10 ft/sec. This corresponds to a speed of about 20 mph!

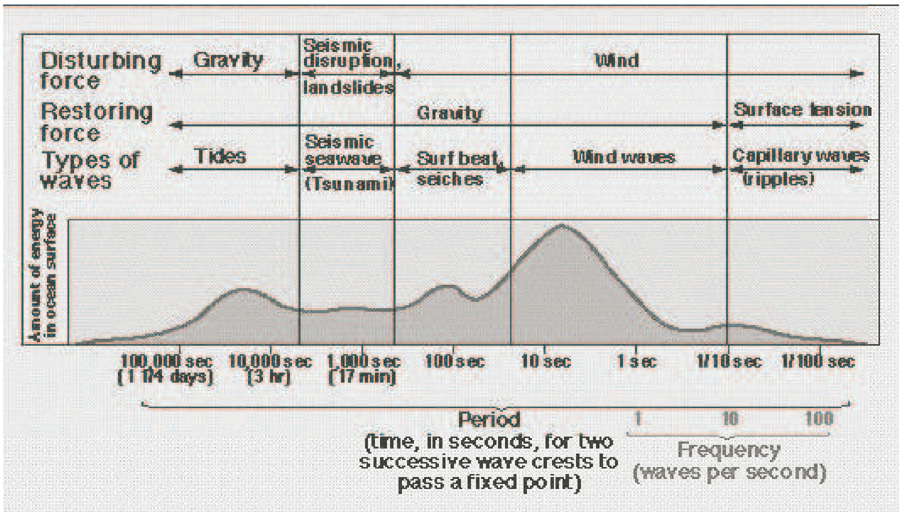

Figure 21: The distribution of wavelengths found on the open ocean

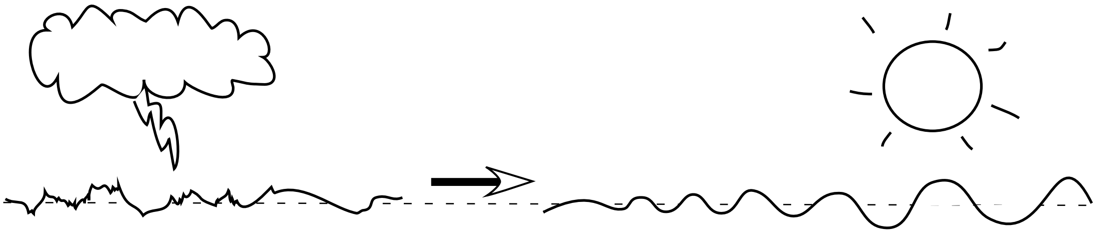

Figure 22: Stormy surf on the open ocean can lead to well organized fetch further away from the storm because of the dispersion relation.

6.1 Gravity Waves

Gravity waves are the waves cause by a vertical perturbation on the surface of a fluids in which the gravitational force responds by trying to reestablish equilibrium. Several assumptions are made in considering gravity waves. They include: incompresibility, irrotational flow, inviscid (no viscosity), a two dimensional wave field, no ambient velocity (current) and small amplitude. After we write down the equations, we will then consider the propagation of waves for three different situations 1) when the depth is much greater than the wavelength, when the depth is comparable and then when the depth is much less than the wavelength[9].

Like any other field in which waves are the main objective to understand, we will seek the dispersion relation. As discussed previously, the dispersion relation relates wavelength to frequency by a proportionality of the the velocity. Knowing the dispersion relation and knowing the wavelength one can calculate the velocity.

We intend to examine small amplitude plane waves. That is, waves that are small with respect to the depth of the water and have small water surface angles. They are sometimes referred to as Stokes waves. Stokes (1847-1880) assumed that the solution to the Euler equation with the appropriate applied boundary conditions could be expressed by a Fourier series. Airy wave theory uses only the first term of this series.

We will consider two different approaches to solving ocean waves. The first will be to consider "body waves" since we already know that the ocean depth is small compared to the wavelength of the wave. This will be referred to as shallow water waves. In the second derivation, we won't assume this, and we'll start from the Laplace equation and try to solve purely "surface waves.

6.1.1 Shallow Water Waves

We refer ourselves to the figure of a wave in which the position and shape of the wave is given by $/eta(x, t)$. In the case of negligible velocity, we can write down the Euler equation for surface waves. The z component of the equation is:

$$\rho \frac{Dv_z}{Dt} = -\frac{\partial P}{\partial z} - \rho g - \rho F_z \tag{48}$$

while the x component is:

$$\rho \frac{Dv_x}{Dt} = -\frac{\partial P}{\partial x} - \rho F_x \tag{49}$$

These equations are pretty difficult to solve, mainly because they are nonlinear. To get around this problem we cave to rely on the "quasistatic" or hydrodynamic limit. Namely, since v is small, v^2 is even smaller and can be negligible. We can also assume that the change in velocity (especially in the z direction) is pretty small. This is true particularly for low amplitude waves of large wavelengths. In this case we are saying that motion in the z direction is so slow that we can take it to be such that hydrostatic equilibrium is maintained at all times.

Taking the equation for the z direction and assuming that

$$\frac{\partial v_z}{\partial t} \simeq 0 \tag{50}$$

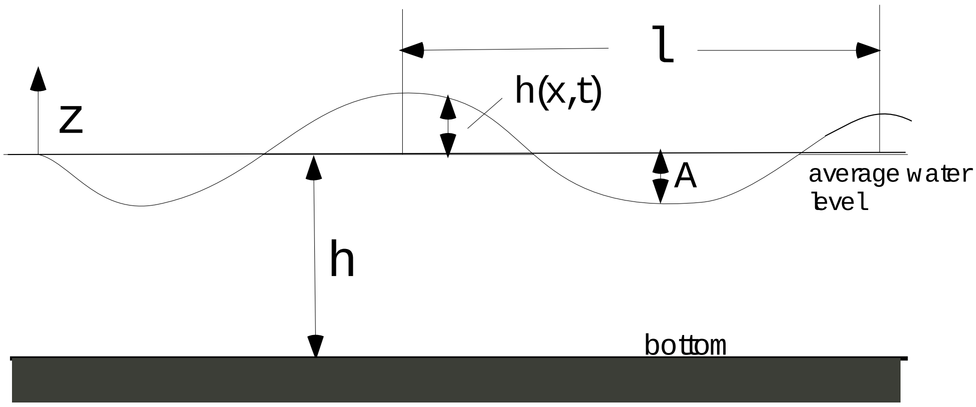

Figure 23: The perturbed surface of a fluid.

the solution to the equation (49) is simply:

$$P - P_o = g\rho(h + \eta - z) \tag{51}$$

Differentiating this equation with respect to x, we get:

$$\frac{\partial P}{\partial x} = g\rho\frac{\partial \eta}{\partial x} \tag{52}$$

which is a reasonable first order approximation that the pressure changing with x is proportional to how fast the surface is changing. Using this result and the initial equation for velocity in the x direction we can write:

$$\frac{\partial v_x}{\partial t} = -g\frac{\partial \eta}{\partial x} + F_x \tag{53}$$

We then go on to consider for an infinitesimal element in the x direction, that the mass of a slice will only change if the level of the surface at that point changes (assuming to a pretty good approximation, incompressibility). The only way mass will change is if the velocity in the x direction changes. Therefore, we write:

$$h\frac{\partial v_x}{\partial x} = \frac{\partial \eta}{\partial t} \tag{54}$$

If we then differentiate this equation with respect to t and the one before that with respect to x, we can eliminate v_x from the equation and write:

$$\frac{\partial^2 \eta}{\partial t^2} = gh\frac{\partial \eta}{\partial x^2} - h\frac{\partial F_x}{\partial x} \tag{55}$$

In the case we have no force except the earth's gravitational field, we are left with the wave equation:

$$\frac{\partial^2 \eta}{\partial t^2} = gh\frac{\partial^2 \eta}{\partial x^2} \tag{56}$$

which has a wavespeed:

$$c = \sqrt{gh} \tag{57}$$

6.1.2 Deep Water Waves

For waves in the deep ocean, we cannot not assume velocity in the vertical direction is negligible. Using the continuity equation and the assumption that the flow is irrotational, we can start with Laplace's equation. Using the velocity potential ϕ

$$v = \nabla\phi \tag{58}$$

we can solve the Laplace equation

$$\nabla^2\phi = 0 \tag{59}$$

The use of this equation already incorporates the equation of continuity, so that the only other equation we need to write down is the Euler equation in terms of the velocity potential. Small amplitude wave theory is based upon the assumptions that all motions are infinitely small which enables us to neglect the square of all the velocity components. Thus we can write:

$$\frac{\partial\phi}{\partial t} = \frac{P}{\rho} + gz \tag{60}$$

Previously we combined the Euler equation with the equation of continuity to display the wave equation explicitly. In this part, we will assume there is a wavelike solution and then verify it.

Let's "guess" that the solution looks like:

$$\phi = f(z)\cos(kx - \omega t) \tag{61}$$

then Laplace's equation reads

$$\frac{d^2 f(z)}{dz^2} - k^2 f(z) = 0 \tag{62}$$

This means that f(z) has to be of the form

$$f(z) = Ae^{kz} + Be^{-kz} \tag{63}$$

At the bottom of the fluid z=0, we know that $v_z = \frac{\partial \phi}{\partial t} = 0$ since no fluid can cross the boundary. This reasons that A=-B so

$$f(z) = 2Acosh(kz) \tag{64}$$

The other boundary condition is that at the surface of the fluid, the pressure is equal to P_o the atmospheric pressure. From this idea we can write:

$$\frac{\partial^2 \phi}{\partial t^2} + g\frac{\partial \phi}{\partial y} = 0 \tag{65}$$

$$\eta = \frac{1}{g}(\frac{\partial \phi}{\partial t}) \tag{66}$$

For small amplitude waves, the vertical velocity component is equal to the rate of rise of the water surface at any point. This can be written as

$$\frac{\partial \eta}{\partial t} = -\frac{\partial \phi}{\partial z} \tag{67}$$

which is known as the kinematic boundary condition. The dynamic boundary condition at the free surface is given by

$$\eta = \frac{1}{g}(\frac{\partial \phi}{\partial t}) \tag{68}$$

Combining these two boundary conditions on the free surfac, we obtain a single free surface boundary condition.

$$\frac{\partial^2\phi}{\partial t^2} + g\frac{\partial\phi}{\partial z} = 0 \tag{69}$$

in this formula we have replaced all quantities which are to be evaluated at the surface $z = h + \eta$ by quantities evaluated at the equilibrium surface z=y.

$$\frac{P}{\rho} + \frac{\partial\phi}{\partial t} + U = 0 \tag{70}$$

The general solution for the velocity potential is

$$\phi = 2A\cosh(kz)\cos(kx - \omega t) \tag{71}$$

If we use the definition of the velocity potential, we can determine the horizontal and vertical components.

$$v_x = -\frac{\partial\phi}{\partial x} \tag{72}$$

$$v_z = -\frac{\partial\phi}{\partial z} \tag{73}$$

The surface displacement is simply(why?)

$$\eta = -\frac{2\omega A}{g}\cosh(kz)\sin(kx - \omega t) \tag{74}$$

6.2 The Long Wavelength Limit

We can use the solutions provided in equation ? and ? to the equation ? will hold provided that:

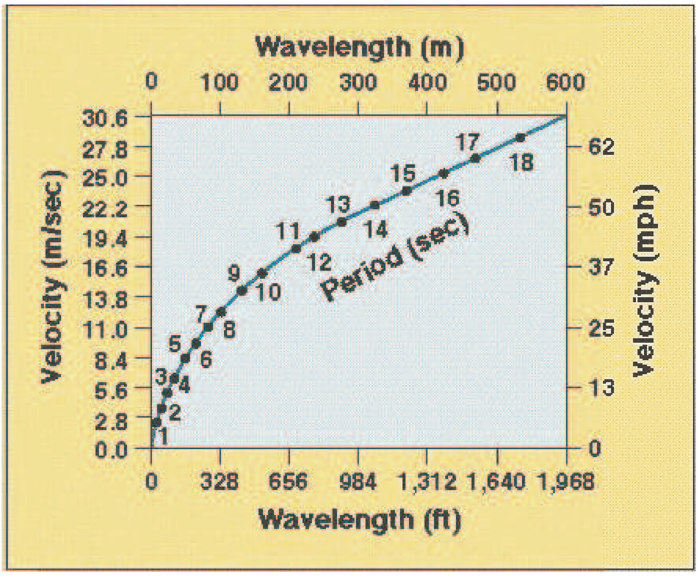

Figure 24: The dispersion relation

$$\omega^2 = gk \tanh(kh) \tag{75}$$

where g is the force of gravity and h is the depth of the water.

We can solve for wave velocity as:

$$c = \frac{\omega}{k} = \sqrt{\frac{g}{k} \tanh(kh)} \tag{76}$$

which can also be written as:

$$c = \sqrt{\frac{g\lambda}{2\pi} \tanh(\frac{2\pi h}{\lambda})} \tag{77}$$

This equation can be rewritten in terms of the wavelength and period

$$\lambda = \frac{gT^2}{2\pi} \tanh(\frac{2\pi h}{\lambda}) \tag{78}$$

When the ocean bottom becomes shallow, tanh(x) x so the "dispersion relation" becomes

$$c = \sqrt{gh} \tag{79}$$

This is the well known equation for the propagation of a small amplitude surge, when the speed becomes independent of the wavelength. This approximation becomes valid when the amplitude of the perturbation A is much smaller than the wavelength λ of the wave it creates. Then the nonlinear terms can be dropped and we are in the "quasistatic limit".

For deep water waves, using the approximation tanh(x) 1, we can write the dispersion relation as:

$$c^2 = \frac{g\lambda}{2\pi} \tag{80}$$

or

$$\lambda = \frac{gT^2}{2\pi} \tag{81}$$

Therefore, on the open ocean where the bottom is very deep, a period of 20 seconds corresponds to a wavelength of about 600 meters!

6.3 The Tides

We will now touch on tidal waves (more commonly referred to as tides). The tides have always played a major role in human affairs. Surfers know the importance that

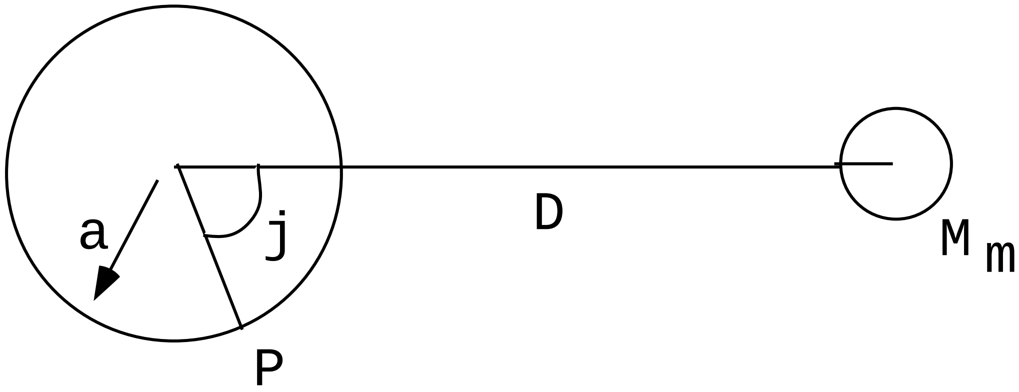

Figure 25: The configuration of the earth and the moon.

the tides play in wave quality.

It is generally known that the tides are caused by effects of the moon's gravitational attraction on the ocean waters. We can write down the net potential that the water on the surface of the ocean feels due to the difference in forces between the earth and the moon. After writing down the potential on the surface of the earth due to both the earth and the moon we can expand the equation out to lowest order. It turns out to be with this approximations

$$U(\Theta) = \frac{Ma^2}{2D^3}(1 - 3\cos^2\Theta) \tag{82}$$

where M is the mass of the moon, a is the radius of the earth and D is the distance from the Earth to the Moon. The angle Θ is the angle a person on the earth's surface makes with the earth moon vector.

As a primary example, we consider tides at the equator. We can use this potential to find the force at a position

$$F(\Theta) = \frac{\partial}{\partial\Theta} U(\Theta) = \frac{3GMa}{2D^3} \sin^2\Theta \tag{83}$$

If we insert this force into the equation for wave height, we can write:

$$\frac{\partial^2\eta}{\partial t^2} = \frac{gh}{a^2}\frac{\partial^2\eta}{\partial\Theta^2} + \frac{3GMa}{2D^3}\cos^2\Theta \tag{84}$$

We will write

$$\Theta = \omega t \tag{85}$$

For the "particular" part of the solution and not the homogeneous term, we will write down

$$\frac{\partial\eta}{\partial t^2} = \frac{3GMah}{D^3(1 - \frac{gh}{\omega^2 a^2})}\cos^2(\omega t) \tag{86}$$

This is easily solved to give

$$\eta = \frac{3GMa^2h}{4D^3(gh - \omega^2 a^2)}\cos^2(\omega t) \tag{87}$$

The first important thing to note is that the water level will reach a maximum (high tide) twice a day (it is diurnal).

The second thing to note is that depending on the sign of $gh - w^2a^2$, the perturbation will attain a maximum or minimum when the moon is directly overhead. since the average depth of the ocean is a few kilometers, the tide will be inverted (we will have a low tide when the moon is directly overhead. The tidal bulge cannot keep up with the moon and it will lag by 180 degrees. This is analogous to a harmonic oscillator driven above resonance frequency. We do indeed know that the tides are semi-diurnal, although the presence of a variable depth in the ocean and land masses complicates the calculation of real tides considerably.

The inertial effects of the ocean that lead to time lag in reference to the moon lead to many interesting results. These effects include large tide swings in interesting geometries such as bays, coves and river mouths.

Although we have always referred to tides as being caused by the moon, every body is capable of exerting a gravitational force on the earths surface. In fact, we can calculate the effects of the tides from the sun, but we don't need to do that now! The corriolis force also affects the movement of the earths ocean but we don't need to get into that now!.

6.4 Capillary Waves

Capillary waves are primarily influenced by strong curvatures of the fluid surface doing work on the surface tension. Capillary waves stretch the surface of the fluid.

We write the "work" dW required to increase the area of the surface by an amount dS.

$$dW = TdS \tag{88}$$

where T is the surface tension of a given interface. In a fluid there is a net inward force due to the attraction of molecules in a fluid. The change in pressure across a curved boundary having two radii's of curvature is:

$$\triangle P = T(\frac{1}{R_1} + \frac{1}{R_2}) \tag{89}$$

Therefore, the smaller the radius of curvature, the larger the surface force. This new force introduces a rather different problem at the surface. Up to this point we have always used the condition that a surface was characterized by a constant value of pressure. But the existence of a force in the surface that could balance a force due to the imbalance in pressure means that we must be more careful. The curvature can be defined as:

$$\frac{1}{R} = \frac{\frac{\partial^2 f}{\partial x^2}}{\sqrt{1 + (\frac{\partial f}{\partial x})^2}} \tag{90}$$

which for small derivatives becomes:

$$\frac{1}{R} \approx \frac{\partial^2 f}{\partial x^2} \tag{91}$$

To solve this type of situation, we can write down the Laplace equation for the velocity potential and the Euler equation for each side of the boundary (both the water and the air). We can use the boundary conditions in this situation including assuming that an element in the surface move at the same velocity of the surface itself. We can then write down:

$$\frac{P_{w,a}}{\rho_{w,a}} = (-\frac{\omega^2}{k} - g)A\sin(kx - \omega t) \tag{92}$$

Using these definitions of pressure and relating them to surface tension, going through a little math, we can write down:

$$\omega^2 = gk\frac{\rho_w - \rho_a}{\rho_w + \rho_a} + \frac{Tk^3}{\rho_w + \rho_a} \tag{93}$$

If we recall that the velocity of a wave is given by the angular frequency divided by the wave vector, we can write:

$$c^2 = \frac{\rho_w - \rho_a}{\rho_w + \rho_a}\frac{g}{k} + \frac{Tk}{\rho_w + \rho_a} \tag{94}$$

The dispersion relation for capillary waves is

$$c^2 = (\frac{\rho_w - \rho_a}{\rho_w + \rho_a})\frac{g\lambda}{2\pi} + \frac{T}{\rho_w + \rho_a}\frac{2\pi}{\lambda} \tag{95}$$

for $\rho_{w,a}$ being the respective densities at the interface. As a useful approximation for long wavelenth, the air density ρ_1 and T can be set to zero. The dispersion relation then reduces to

$$c = \sqrt{\frac{g}{k}} \tag{96}$$

This is a nice sanity check, that we reduce to the long wavelength limit for situations in which the bottom is very (infinetly) deep and surface tension is negligible.

For large wavelengths, while still considering the difference between densities as important, the equation can be approximated as:

$$c = \sqrt{(\frac{\rho_1 - \rho_2}{\rho_1 + \rho_2})\frac{g\lambda}{2\pi}} \tag{97}$$

and for small wavelengths it becomes:

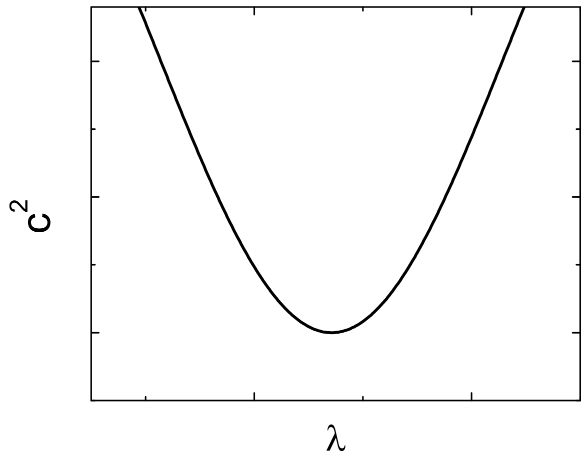

Figure 26: The dispersion relation for capillary waves showing the minimum speed for a certain interface

$$c = \sqrt{\frac{T}{\rho_1 + \rho_2} \frac{2\pi}{\lambda}} \tag{98}$$

Capillary waves have a minimum speed at

$$\lambda = 2\pi \sqrt{\frac{T}{g(\rho_1 - \rho_2)}} \tag{99}$$

We can see here that for smaller wavelengths, the wave speed diverges. Is this physical? May it be physical only down to the atomic scale of a fluid in which band gaps become reasonable.

We can retrace these steps for when the upper medium is moving with a velocity U with respect to the lower medium which is clearly a situation for waves generated by the wind

We write

$$c = \frac{U\rho_a}{\rho_a + \rho_w} \pm [(\frac{\rho_w - \rho_a}{\rho_w + \rho_a})\frac{g\lambda}{2\pi} + \frac{T}{\rho_w + \rho_a}\frac{2\pi}{\lambda} + \frac{U^2\rho_a\rho_w}{\rho_w + \rho_a}] \qquad (100)$$

6.5 Tsunamis

Tsunamis are very long wavelength water waves created by underwater earthquakes and volcanic eruptions. Even where the water is deep, say 3 miles, the wavelength is usually comparable, making them behave as a shallow water wave. From this depth, we can calculate that the mean velocity is nearly 500 miles/hour! As a tsunami approaches shore, the small, almost undetectable disturbance rapidly grows. It is capable of producing catastrophic damage and substantial loss of life[10].

7 The Ocean Meets the Land

A wave of the sea always breaks in front of its base and that portion of the crest will then be the lowest of which before was the highest.

-Leonardo DaVinci (Notebooks Vol 2)

7.1 Island Shadowing

The process of wave refraction and diffraction lead to changes in wave height along the crest. Numerical modeling can demonstrate how large scale bathymetric structures such as ridges and canyons result in wave height variation along the coast with beneficial or detrimental consequences for wave quality and surfing conditions.

7.1.1 Shoaling

So far we have been concerned with the behavior of regular waves propagating over a smooth horizontal sea-bed in the absence of variation in water depth, the presence of current or obstacles in the flow. However, as a "wave train" propagates into shallow water, a change in wave height and wavelength can be observed. This process is referred to as "shoaling". The solution to the complete boundary value problem in which the boundary condition at the sea bed takes into consideration the variation in depth is very difficult (impossible?) to solve analytically. There are numerical techniques that operate well under the assumptions that the motion is two dimensional, the period is constant and the average energy transfer in the direction of the wave propagation is also constant. These assumptions require that

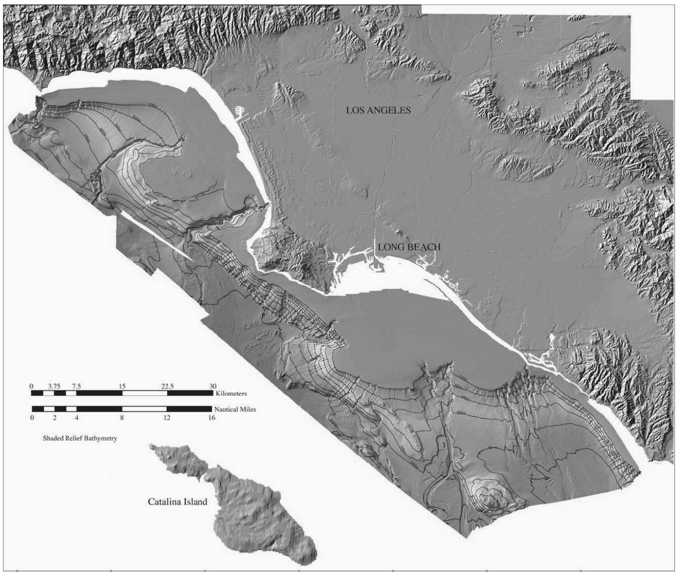

Figure 27: Some topology of the ocean bottom near the southern california coast. The shallow bottom substantially extending outwards near long beach cause wave heights to be substantially lower than its souther neighbor, Huntington Beach. The underwater canyons along the Los Angeles coastline are often major factors in determining wave conditions.

Figure 28: A great example of shoaling and refraction.

the sea bed has a gentle slope such that wave reflection can be neglected and that the wave is neither enhanced nor dissipated by the bottom friction.

7.1.2 Refraction

It is observed in the ocean when waves approach a bottom slope obliquely, the speed of the wave in shallower water is much smaller than in deeper water. As a result, the line of the wave crest is bent so as to become more closely aligned with the bottom contours. This is commonly known as wave refraction.

Wave period is assumed constant so:

$$\omega = C_1 k_1 = C_2 k_2 \tag{101}$$

When approaching an abrupt contour change in the ocean bottom, the waves

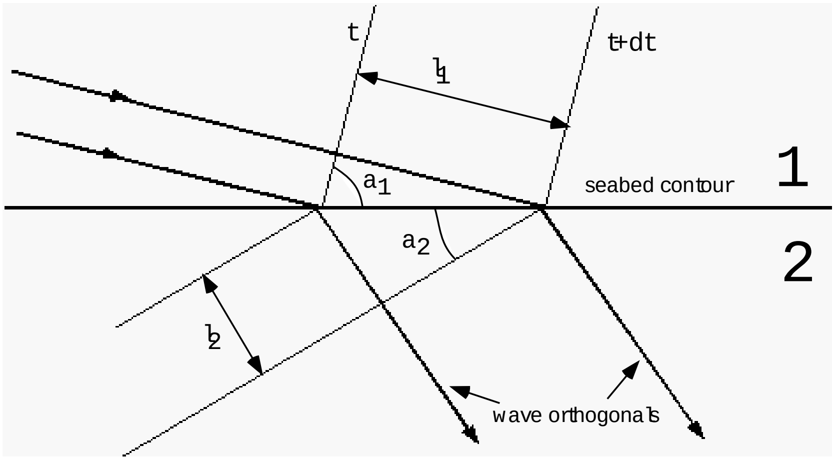

Figure 29: Snells law. As the wave approaches a boundary change (discrete or continuous) the waves partially transmits through at a new angle and partially reflects.

change their direction. If the wave speed changes from C1 to C2 we write

$$\frac{C_1}{C_2} = \frac{\sin(\alpha_1)}{\sin(\alpha_2)} \tag{102}$$

In more general, for a continuously changing ocean bottom we can write:

$$\frac{C}{C_o} = \frac{\sin(\alpha)}{\sin(\alpha_o)} \tag{103}$$

Likewise we can write down the ray equations and may be solved to determine the variation of α and the path of orthogonals.

7.1.3 Diffraction

When a wave train encounters a large vertical obstacle, it has been observed that the wave motion penetrates into the region of the "geometric shadow".

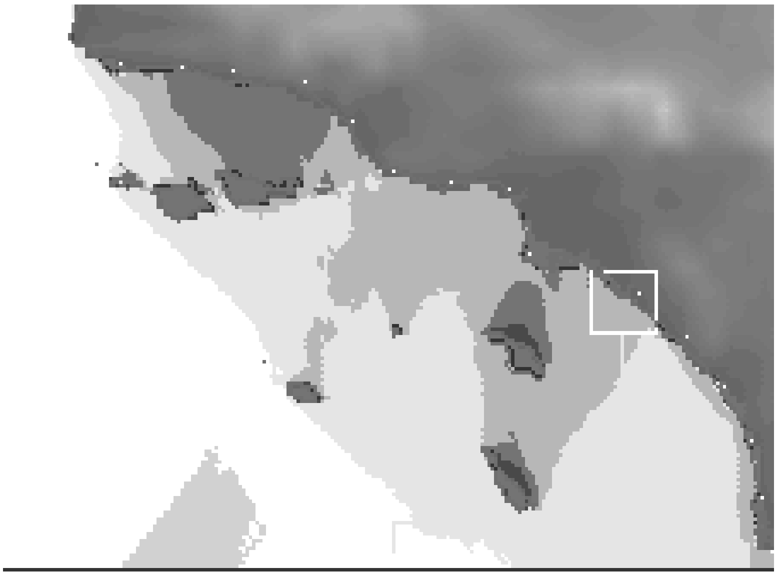

Figure 30: The Channel Islands offshore create shadowing for the waves that get to the mainland.

We can write the energy corresponding to a single wavelength are of the form

$$E = \frac{\rho g A^2}{8} \tag{104}$$

The energy flux is defined as

$$P = EC_g \tag{105}$$

where C_g is the group velocity. We can write down the rate of energy transfer, P, of a wave

$$P = \frac{1}{2}\rho g A^2 C_g \tag{106}$$

Considering the velocity potential, we say for a wave train encountering an obstacle we write:

$$\phi = \phi_{incident} + \phi_{scattered} \tag{107}$$

There are three situations of dealing with the scattering of waves.

The first is the Morrison equation which is used when the size of a structure is small compared with the water wavelength. It is composed of two forces: inertia and drag forces. The second is the Froude Krylov theory which is used when the drag force is small in comparison with inertia but the size of the structure is still relatively small. The third situation is the simple theory of diffraction. It is used when the size of the structure is large (spanning a significant fraction of the wavelength) the incomming waves undergo significant scattering (diffraction) after striking it. Solutions generally involve the Green's function method, finite element method of boundary element method to solve.

7.2 Dynamics of Breaking Waves

When a wave breaks the energy found in the bending of the ocean's surface is transferred elsewhere, first into potential energy (the wave height increases) then into sound, heat, erosion etc. We can say that there are generally two types of waves: spilling (crumbler) waves and plunging (spitting, throwing, dumping) waves. The difference between the two generally depends on the angle of the seabed floor. For a critical angle, the wave can transition from a crumbler to a spitter. There are considered four main parameters to a breaking wave which include wave height,

Figure 31: The wedge in Newport beach, California is a very interesting breaking wave.

intensity, peel angle, and wave section length. The peel angle is a critical term used to describe the speed that a surfer needs to successfully traverse across the face of the wave. It is defined as the angle between the trail of the broken white water and the crest of the unbroken part as it propagates forward. The angle ranges between 0 and 90 with small peel angles resulting in fast surfing waves and large angles in slow waves. There is a limit to how small a peel angle can be before it is impossible to surf it. A further mechanism affecting the form of a breaking wave due to irregularities in local bathymetry are partial wave reflections from the front steps in the reef face.

It is important to understand how breakwaters and other engineered offshore structures take the energy from incoming waves. For very small amplitude waves, breakwaters can be modelled as an impenetrable boundary in which the laws of reflection for linear waves hold reasonably well. For any larger waves, nonlinearity become a difficult problem[11].

7.3 Nonlinear Waves: Solitons

In previous sections, small amplitude waves were assumed. There the free surface boundary conditions were linearized by assuming the the contribution of second and higher order terms was negligible. However, much experimental evidence indicates that large waves are highly nonlinear. Nonlinear studies is a still emerging field that yields exciting new theoretical discoveries in a variety of other fields besides ocean waves. Nonlinearity provides the emergence of new structures that are spatially or temporally coherent. Such emergent structures have their own features, their own lifetimes and peculiar ways of interacting.

The field might have said to have began one late summer day in 1834, when John Scott Russel, a young Scotish engineer recognized a localized structure as a water wave and followed it along a canal that did not disperse for over a mile. No one believed his story for years following a mathematical proof of such a physical phenomenon did not emerge until years later. Diedrerik Kortweg and Henderick de Vries published a theory on shallow water waves later on and they found that solutions to their nonlinear equation (KdV): were also non dispersive and traveled at a fixed velocity! They noted that the dispersive term could be "balanced" or counteracted by the nonlinear term. This equation is now used whenever one studies the unidiretional propagation of long waves in a dispersive, energy-conserving medium. The appropriatte amplitude equations for traveling wave systems near the onset of instability are the extensions of the amplitude equation for stationary systems in which the coefficients of the various terms are complex (the imaginary parts of these coefficients are associated with the change in frequency of the waves with wavenumbers and with the amplitude of the mode. These equations are often referred to as the complex Ginzburg Landau equations.

Most discussions on nonlinear waves and solitons start with a quotation from J. Scott Russell's 'Report on Waves' in 1844 describing his famous chase on horseback behind a wave in a channel.

I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped-not so the mass of water in the channel which it had put in the motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity assuming the form of a large solitary elevation, a rounded, smooth and well defined heap of water which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling at a rate of some eight to nine miles and hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August, 1834, was my first chance interview with that singular and beautiful phenomenon.

The Nonlinear Schrodinger Equation (NLSE) :

$$i\frac{\partial \psi}{\partial t} = \nabla^2 \psi + V(x)\psi + \|\psi^2\|\psi \tag{108}$$

is a useful and generic equation that is used when one wishes to consider the first

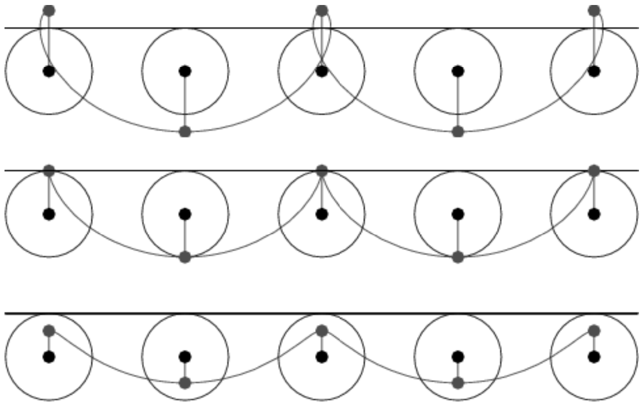

Figure 32: Place a point on a wheel that trace out the path it makes as the wheel rolls along the surface. Depending on where the point is in relation to how the wheel contacts the surface determines the shape of the wave that you get. These functions are fairly successful in describing the shape of a wave near the point when it begins to break.

effects of dispersion and nonlinearity on a wave packet. It is applied to nonlinear optics and filamentation, a term use to describe the localization of things such as the paths taken by streams, rivers, lightening and even animal trails. There has been extensive work using the Discrete Nonlinear Schrodinger Equation (DNLSE) to one and two dimensional lattices. The dispersive term considered is usually the interaction term Vn, which can describe nearest neighbor interactions as well long range effects.

The standard soliton solution is:

$$\eta = sech^2(\frac{\sqrt{3}}{2}\sqrt{\frac{A}{h^3}}(x - ct)) \tag{109}$$

There is a common solution to waves in a shallow medium. The shape of the wave can be defined by a trochoidal function. A trochoidal function is mapped about by tracing the path that a point on a sphere would take as it is rolled along a flat surface. The x and y coordinates are defined by:

$$x = a\phi - b\sin\phi \tag{110}$$

$$y = a - b\cos\phi \tag{111}$$

Ocean waves begin as "curtate" when a¡b. As they begin to peak and break they become prolate for b¿a.

8 Man Made Waves

With the knowledge of the past sections about wave propagation on the open seas and the interaction with land, we can think of what possibilities there are to create man made structures that influence how waves break. Such knowldege can be used for example, to fine tune the dynamics of a wave making it ideal for a certain surfing style. The growing passion of people around the world to ride waves make man made surfing structures a mainstream consideration.

8.1 Wave forecasting

Wave Forecasting has come a long way in recent years. Satellite and buoy data along with accurate modelling of the ocean can predict swells sometimes over a week in advance. One of the greatest examples of surf forecasting is a computer model called Lola developed by forecasters at Surfline [12]. Using detailed knowledge of ocean currents and ocean bottom topology, LOLA has proven to be the most powerful wave forecasting system used by surfers today.

8.2 Wave Pools

Surfing in wave pools has become a popular new trend. All over the world, blueprints for wave pools and construction plans are being considered. The problem of artificially making waves that look, break and are surf realistically is not easy. Contractors combine duplication of real surfing reef characteristics, computer design optimization and large scale laboratory basins to create custom designs with in-

Figure 33: Currently the best artificial wave in the world. We should be able to do alot better.

creasingly successful results.

Interest in surfing pools goes back to the construction of the worlds first surfing pool at Big Surf at Tempe, Arizona completed in 1969. Surfing pools are different to the normal wavepool as they attempt to produce waves of greater height and better quality to the interest of surfers.

A surfing pool, designed by the late Surf Tech Ltd of California was built at Dorney Park, Allentown, Pennsylvania and was used to hold the world's first "inland professional surfing championship" on June 23, 1985. however, this surfing pool also produces extremely small and weak waves and the event was a dismal failure in terms of wave quality. In fact the Association of Surfing Professionals threatened to ban competitions at wave pools until standard wave criteria were met. There is still a great potential for improvement in the design of surfing pools. Still, a true surfers pool has never been built.

The most successful wavepool to date has been constructed in Orlando Florida.

It is called Typhoon Lagoon. It uses a self interference effect by sending waves at an angle to a wall. As the reflect off the wall, they constructively interfere with themself, causing the wave to peak and peel of the wall.

Waveparks designed to be built in the future claim 1 dollar fees for every wave caught.

8.3 Artificial Reefs

A variety of aspects of artificial reef construction have been considered including reefs relation to sediment transport and erosion as well as the topological shape of surfing breaks. There has been two "Artificial Surfing Reef Symposiums" held so far, the first being in Sydney Australia in 1997 and the second held in San Diego California in 1998.

Pratte's reef in El Segundo California was constructed on September 22 2000 out of 1600 cubic meters of sand contained in 200 geotextile bags. They were thrown into the surf in a V shape with the apex pointing offshore. Over the first two years that it was constructed there have been about 10 surfable sessions at the reef. These sessions generally referred to shorter period swells, as the larger period swells were too big for the reef to be well affected. Currently, Pratte's reef is no longer affecting incoming swells in any way. The reef bags are almost level with the sand. The overall poor performance of Pratte's Reef as a surf spot can be largely attributed to deficiencies in the design rather than other factors such as poor location or a deficient wave climate.

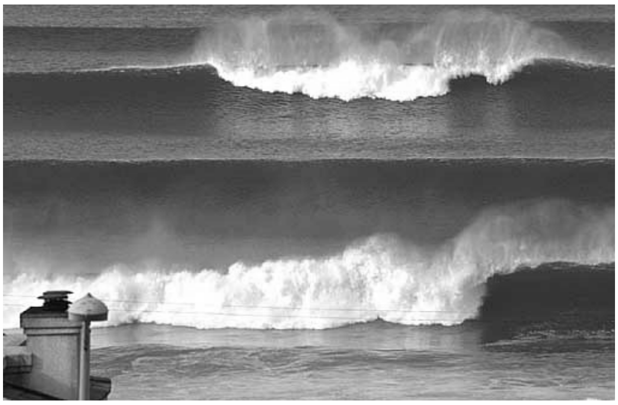

Figure 34: One of the best days of surf El Porto ever had

The incorporation of wave focusing features in artificial reef construction can greatly improve the quality of a surfing break.

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